

Symplectic embeddings of four-dimensional polydisks into balls

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Abstract

In this paper we obtain new obstructions to symplectic embeddings of the four-dimensional polydisk $P(a, 1)$ into the ball $B(c)$ for $2 \leq a < \frac{\sqrt{7}-1}{\sqrt{7}-2} \approx 2.549$, extending work done by Hind-Lisi and Hutchings. Schlenk’s folding construction permits us to conclude our bound on c is optimal. Our proof makes use of the combinatorial criterion necessary for one “convex toric domain” to symplectically embed into another introduced by Hutchings in [8]. Additionally, we prove that if certain symplectic embeddings of four dimensional convex toric domains exist then a modified version of this criterion from [8] must hold, thereby reducing the computational complexity of the original criterion from $O(2^n)$ to $O(n^2)$.

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1 Introduction

1.1 New obstructions to embeddings of four-dimensional polydisks

In this paper we investigate the question of when one convex toric symplectic four-manifold with boundary can be symplectically embedded into another. In particular, we obtain new sharp obstructions to symplectic embeddings of the four-dimensional polydisk $P(a, 1)$ into the ball $B(c)$. In addition, we reduce the computational complexity of obstructing symplectic embeddings of convex toric four manifolds. Toric manifolds are defined follows and they admit a symplectic structure coming from $\omega_{\mathbb{C}^2}$.

Definition 1.1. Let Ω be a polygonal domain in the first quadrant of \mathbb{R}^2 . Then, we associate to Ω a subset X_Ω of \mathbb{C}^2 defined by

$$X_\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \mid (\pi|z_1|^2, \pi|z_2|^2) \in \Omega\}.$$

X_Ω is a symplectic submanifold of \mathbb{C}^2 , with symplectic form given by the restriction of the standard form on \mathbb{C}^2 , namely

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

We call X_Ω the *toric domain* associated to Ω . Suppose that Ω is of the form

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq A, 0 \leq y \leq f(x)\},$$

where $f : [0, A] \rightarrow \mathbb{R}_{\geq 0}$ is a nonincreasing function. If f is concave, then we say that X_Ω is a *convex toric domain*. If f is convex, then we say that X_Ω is a *concave toric domain*.

Example. Let Ω be the triangle in \mathbb{R}^2 with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$ for any $a, b > 0$. Then, X_Ω is the 4-dimensional ellipsoid

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1 \right\}.$$

When $a = b$, X_Ω is the 4-dimensional ball $B(a) = E(a, a)$. The ellipsoid $E(a, b)$ is both a concave and a convex toric domain, since Ω is the region lying beneath the line $f(x) = (-b/a)x + b$ in the first quadrant of \mathbb{R}^2 .

Example. Let Ω be the rectangle in \mathbb{R}^2 with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) for any $a, b > 0$. Then, X_Ω is the polydisk

$$P(a, b) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \leq a, \pi|z_2|^2 \leq b\}.$$

The polydisk $P(a, b)$ is a convex toric domain, since Ω is the region lying beneath the constant function $f(x) = b$ on the interval $[0, a]$.

The question of when one toric domain symplectically embeds into another is not generally understood, but progress has been made for certain cases, primarily in dimension 4. In [7], Hutchings associates to any symplectic four-manifold with boundary X a sequence of real numbers,

$$0 = c_0(X) \leq c_1(x) \leq c_2(x) \leq \dots,$$

such that if X symplectically embeds into X' , then

$$c_k(X) \leq c_k(X') \text{ for all } k.$$

The c_k are called *ECH capacities* (here ECH stands for “embedded contact homology,” which Hutchings uses to define the capacities). Work by Choi, Cristofaro-Gardiner, Frenkel, Hutchings, and Ramos [2], computed the ECH capacities of all concave toric domains, yielding sharp obstructions in many interesting cases. Cristofaro-Gardiner [3] showed that ECH capacities give sharp obstructions to symplectic embeddings of any concave toric domain into any convex toric domain. His result generalizes the well-known results of McDuff [9]–[10] and Frenkel–Müller [4].

However, obstructions via ECH capacities are suboptimal in the case of symplectic embeddings of a convex toric domain into a concave toric domain. For instance, using the ECH capacities of polydisks and balls (which Hutchings explicitly computes in [7]), one can only show that the obstruction to symplectically embedding $P(2, 1)$ into $B(c)$ must satisfy $c \geq 2$. On the other hand, a result due to Hind and Lisi [5] indicates that $P(2, 1)$ does not symplectically embed into $B(c)$ for any $c < 3$.

For this reason, Hutchings studied embedded contact homology in a more refined way than is used to define the ECH capacities. As a result, he was able to give a new combinatorial criterion for obstructing symplectic embeddings, [8, Theorem 1.19], which we will hereafter term *the Hutchings criterion*. The Hutchings criterion is a somewhat complicated combinatorial condition; we will defer a full description of it to the next section. Hutchings used this criterion to demonstrate several new bounds on embeddings of polydisks into balls, ellipsoids, and polydisks.

Our first result is the following extension of Hutchings [8, Theorem 1.4] and Hind–Lisi [5] on symplectic embeddings of polydisks into balls.

Theorem 1.2. *Let*

$$2 \leq a < \frac{\sqrt{7} - 1}{\sqrt{7} - 2} = 2.54858\dots$$

If $P(a, 1)$ symplectically embeds into $B(c)$ then

$$c \geq 2 + \frac{a}{2}.$$

Remark 1.3. The bound on c in this theorem is optimal: in [11, Prop. 4.3.9], Schlenk uses “symplectic folding” to construct a symplectic embedding $P(a, 1) \hookrightarrow B(c)$ whenever $a > 2$ and $c > 2 + a/2$.

Remark 1.4. The first result of this nature was due to Hind and Lisi, who proved that $P(2, 1)$ does not symplectically embed into $B(c)$ for any $c < 3$. Hutchings then proved the statement of Theorem 1.2 for $2 \leq a \leq 2.4$ using the Hutchings criterion and conjectured that the full statement of Theorem 1.2 could be proven using the Hutchings criterion [6]. Our proof thus answers this conjecture in the affirmative.

The proof of Theorem 1.2 can be found in Section 3. In Sections 3.4 and 3.5, we show that it is impossible to extend the upper bound on a in Theorem 1.2 using the current statement of the Hutchings criterion. The conjectural improvement of the Hutchings criterion [8, Conj. A.3], proven in [1], allows for some possibility to extend this upper bound on a . However, such an extension seems unlikely even with the conjectured improvement. For more information, see Section 3.5. Moreover, for $a > 4$, it is known that one can do better than $c > 2 + a/2$; see [11, Fig. 7.2].

Our other two results pertain to the technical details of the Hutchings criterion. The first is a general criterion that can help obstruct many scenarios of interest when applying the Hutchings criterion. This result is instrumental to our proof of Theorem 1.2. The second yields a combinatorial simplification of the Hutchings criterion for obstructing symplectic embeddings. This reduces the amount of computations needed to verify the existence of obstructions from $O(2^n)$ to $O(n^2)$. We state both results in the Section 1.3 after reviewing the necessary background.

1.2 Review of convex generators

We begin by defining the principal combinatorial objects involved in stating the Hutchings criterion. Our exposition closely follows [8, Section 1.3].

Definition 1.5. A *convex integral path* Λ is a path in \mathbb{R}^2 such that:

- The endpoints of Λ are $(0, y(\Lambda))$ and $(x(\Lambda), 0)$ for some non-negative integers $x(\Lambda)$ and $y(\Lambda)$;
- The path Λ is the graph of a piecewise linear concave function $f : [0, x(\Lambda)] \rightarrow [0, y(\Lambda)]$ with $f'(0) = 0$, possibly together with a vertical line segment at the right;
- The vertices of Λ (i.e. the points at which its slope changes) are lattice points.

Definition 1.6. A *convex generator* is a convex integral path Λ such that:

- Each edge of Λ (i.e. the line segment between two vertices) is labelled e or h ;
- Horizontal and vertical edges can only be labelled e .

Because we will work with convex generators frequently, we require a compact notation for them. For any nonnegative, coprime integers a and b and any positive integer m , we will denote by $e_{a,b}^m$ an edge of a convex generator that is labelled e and has displacement vector $(ma, -mb)$. Similarly, $h_{a,b}$ denotes an edge labelled h that has displacement vector $(a, -b)$, while $e_{a,b}^{m-1}h_{a,b}$ denotes an edge labelled h that has displacement vector $(ma, -mb)$. Since a convex generator is uniquely specified by the set of its edges, this notation provides an equivalence between a convex generator and a commutative formal product of symbols $e_{a,b}$ and $h_{a,b}$, where no two distinct factors $h_{a,b}$ and $h_{c,d}$ have $a = c$ and $b = d$ and where there are no factors of $h_{1,0}$ or $h_{0,1}$.

As explained in [8, §6], the boundary of any convex toric domain can be perturbed so that for its induced contact form and up to large action, the ECH generators correspond to these convex generators. Before continuing to draw parallels with ECH, we first describe a few useful aspects of convex generators.

Definition 1.7. Let Λ_1 and Λ_2 be convex generators. Then, we say that Λ_1 and Λ_2 *have no elliptic orbit in common* if, when we write out Λ_1 and Λ_2 as formal products, no factor of $e_{a,b}$ appears in both Λ_1 and Λ_2 . Likewise, we say that Λ_1 and Λ_2 *have no hyperbolic orbit in common* if, when we write out Λ_1 and Λ_2 as formal products, no factor of $h_{a,b}$ appears in both Λ_1 and Λ_2 .

If Λ_1 and Λ_2 are convex generators with no hyperbolic orbit in common, then we define the *product* $\Lambda_1 \cdot \Lambda_2$ to be the convex generator obtained by concatenating the formal product expressions of Λ_1 and Λ_2 (and sorting the factors by slope to ensure that the resulting path is convex). This product operation is associative whenever it is defined.

There are several combinatorial quantities associated to Λ that will be of interest to us.

Definition 1.8. Let Λ be any convex generator.

1. The quantity $L(\Lambda)$ is the number of *lattice points interior to and on the boundary* of the region bounded by Λ and the x - and y -axes.
2. The quantity $m(\Lambda)$ is the *total multiplicity* of all the edges of Λ , i.e. the total exponent of all factors of $e_{a,b}$ and $h_{a,b}$ in the formal product for Λ . Note that $m(\Lambda)$ is equal to one less than the number of lattice points on the path Λ .
3. The quantity $h(\Lambda)$ is the number of edges of Λ labelled h .

Remarkably, one can actually express the ECH index in terms of the above combinatorial data associated to convex generators.

Definition 1.9. If Λ is a convex generator, define the *ECH index* of Λ to be

$$I(\Lambda) = 2(L(\Lambda) - 1) - h(\Lambda).$$

Definition 1.10. Let Λ be a convex generator, and let X_Ω be a convex toric domain. We define the *symplectic action* of Λ with respect to X_Ω by

$$A_\Omega(\Lambda) = A_{X_\Omega}(\Lambda) = \sum_{\nu \in \text{Edges}(\Lambda)} \vec{\nu} \times p_{\Omega, \nu}.$$

Here, for any edge ν of Λ , $\vec{\nu}$ denotes the displacement vector of ν , and $p_{\Omega, \nu}$ denotes any point on the line ℓ parallel to $\vec{\nu}$ and tangent to $\partial\Omega$. Tangency means that ℓ touches $\partial\Omega$ and that Ω lies entirely in one closed half plane bounded by ℓ . Moreover, ‘ \times ’ denotes the determinant of the matrix whose columns are given by the two vectors.

Next, we compute the symplectic action of any convex generator associated to our top most favorite toric domains.

Example.

- If $X_\Omega = P(a, b)$ is a polydisk, then for any convex generator Λ ,

$$A_{P(a, b)}(\Lambda) = bx(\Lambda) + ay(\Lambda).$$

- If $X_\Omega = E(a, b)$ is an ellipsoid, then for any convex generator Λ , then $A_{E(a, b)}(\Lambda) = c$, where the line $bx + ay = c$ is tangent to Λ at some point.

We have yet another definition, which is integral to the combinatorial means of computing ECH capacities as follows.

Definition 1.11. Let X_Ω be a convex toric domain. We say that a convex generator Λ with $I(\Lambda) = 2k$ for some integer k is *minimal* for X_Ω if:

- All edges of Λ are labelled e ;
- For any other convex generator Λ' with all edges labelled e such that $I(\Lambda') = 2k$, we have

$$A_\Omega(\Lambda) < A_\Omega(\Lambda').$$

The symplectic action of minimal generators is related to the ECH capacities as follows.

Remark 1.12. By [8, Prop 5.6] if $I(\Lambda) = 2k$ and Λ is minimal for X_Ω then $A_\Omega(\Lambda) = c_k(X_\Omega)$.

Our final definition will be key to the understanding when one convex toric domain can be symplectically embedded into another convex toric domain.

Definition 1.13. Let X_Ω and $X_{\Omega'}$ be convex toric domains, and let Λ and Λ' be convex generators. We write $\Lambda \leq_{X_\Omega, X_{\Omega'}} \Lambda'$ or $\Lambda \leq_{\Omega, \Omega'} \Lambda'$ if

- (1) $I(\Lambda) = I(\Lambda')$,
- (2) $A_\Omega(\Lambda) \leq A_{\Omega'}(\Lambda')$, and
- (3) $x(\Lambda) + y(\Lambda) - \frac{h(\Lambda)}{2} \geq x(\Lambda') + y(\Lambda') + m(\Lambda') - 1$.

In particular, if X_Ω symplectically embeds into $X_{\Omega'}$, then the resulting cobordism between their (perturbed) boundaries implies that $\Lambda \leq_{X_\Omega, X_{\Omega'}} \Lambda'$ is a necessary condition for the existence of an embedded irreducible holomorphic curve with ECH index zero between the ECH generators corresponding to Λ and Λ' . The inequality (3) is what ultimately allowed Hutchings to go “beyond” ECH capacities. It emerges from the fact that every holomorphic curve must have nonnegative genus [8, Prop 3.2].

We now have all the ingredients needed to state the Hutchings criterion and our modification.

1.3 A modification of the Hutchings criterion

The statement of the criterion we use to obstruct symplectic embeddings will be very similar to the one given by Hutchings in [8, Thm 1.19]. Our modification reduces the amount of computation required to check the criterion.

Theorem 1.14 (The Modified Hutchings criterion). *Let X_Ω and $X_{\Omega'}$ be convex toric domains, and let Λ' be a minimal generator for $X_{\Omega'}$. Suppose that X_Ω symplectically embeds into $X_{\Omega'}$. Then, there exists a convex generator Λ , a nonnegative integer n , and factorizations $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$ such that:*

- (i) *For all i , $\Lambda_i \leq_{\Omega, \Omega'} \Lambda'_i$;*
- (ii) *For all $i \neq j$, if $\Lambda'_i \neq \Lambda'_j$ or $\Lambda_i \neq \Lambda_j$, then Λ_i and Λ_j have no elliptic orbit in common; and*
- (iii) *For all $i \neq j$, we have $I(\Lambda_i \cdot \Lambda_j) = I(\Lambda'_i \cdot \Lambda'_j)$.*

Remark 1.15. The difference between Theorem 1.14 and the original Hutchings criterion [8, Thm 1.19] is in the third bullet point, where Hutchings’ formulation reads:

- (iii)' *If S is any subset of $\{1, \dots, n\}$, then $I(\prod_{i \in S} \Lambda_i) = I(\prod_{i \in S} \Lambda'_i)$.*

In regards to obstructing symplectic embeddings, we show that the conditions on indices of products of three or more factors (as in (iii)') do not contribute any information beyond the conditions on indices of single factors and products of two factors (as in (i) and (iii)). That it is sufficient to consider the indices of single factors and indices of products of two factors, i.e. that (i) and (iii) imply (iii)', is the content of our following result.

Theorem 1.16. *Let $\{\Lambda'_i\}_{i=1}^n$ and $\{\Lambda_i\}_{i=1}^n$ be two sets of convex generators such that the Λ'_i have no hyperbolic orbit in common and the Λ_i have no hyperbolic orbit in common. Suppose that for any $1 \leq i \leq n$,*

$$I(\Lambda_i) = I(\Lambda'_i),$$

and moreover that, for any $i \neq j$,

$$I(\Lambda_i \cdot \Lambda_j) = I(\Lambda'_i \cdot \Lambda'_j).$$

Then, for any subset $S \subseteq \{1, 2, \dots, n\}$,

$$I\left(\prod_{i \in S} \Lambda_i\right) = I\left(\prod_{i \in S} \Lambda'_i\right).$$

The proof of Theorem 1.16 is in Section 2.4. It relies on a formula for the index of the product of two convex generators, which we prove in Section 2.3. Theorem 1.14 now directly follows from Theorem 1.16 when paired with the original Hutchings criterion, [8, Thm 1.19].

Remark 1.17. Checking that (iii)' is satisfied requires comparing two indices of convex generators in $O(2^n)$ different scenarios. Checking that (iii) is satisfied requires comparing two indices in $O(n^2)$ different scenarios. This vast reduction in complexity is beneficial in many circumstances.

We end this section by discussing the following proposition, which is useful for understanding the combinatorial ramifications of the modified Hutchings criterion. We will prove this proposition in Section 2.2.

Proposition 1.18. *Let X_Ω and $X_{\Omega'}$ be convex toric domains, and let Λ and Λ' be nontrivial convex generators with no edges labelled 'h.' Suppose that $\Lambda \leq_{\Omega, \Omega'} \Lambda'$ and $I(\Lambda \cdot \Lambda) = I(\Lambda' \cdot \Lambda')$. Then,*

$$8A(\Lambda') \leq (b(\Lambda') - 1)^2.$$

Moreover, Λ must be of the form $e_{x,y}$, where $x, y \in \mathbb{Z}_{>0}$ are coprime and satisfy

$$xy = 2A(\Lambda')$$

and

$$x + y = b(\Lambda') - 1.$$

Equivalently, x and y must be nonnegative coprime integers such that

$$\{x, y\} = \left\{ \frac{b(\Lambda') - 1 \pm \sqrt{(b(\Lambda') - 1)^2 - 8A(\Lambda')}}{2} \right\}. \quad (1)$$

Remark 1.19. There are a few interesting interactions between the conditions of Theorem 1.14 and Proposition 1.18. For instance, in Section 2.2 we explain how this allows us to rewrite (ii) of Theorem 1.14 as:

- (ii) For all $i \neq j$, if Λ_i and Λ_j have any elliptic orbit $e_{x,y}$ in common, then $\Lambda_i = \Lambda_j = e_{x,y}$.

In addition, by arguing as in the proof of Theorem 1.2, one can sometimes use (i) of Theorem 1.14 along with Proposition 1.18 to prove that the set of possible values of $I(\Lambda')$ is bounded. For more explanation on this type of argument, see the beginning of Section 3 as well as Section 3.3.

The implications of Proposition 1.18 with regard to the Hutchings criterion are further discussed in Remark 2.6.

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2 The modified Hutchings criterion via combinatorial gymnastics

Section 2.1 contains the necessary notation and a useful formula for the index of a convex generator. Section 2.2 investigates the implications of repeated factors in the Hutchings criterion. A formula for the index of the product of two convex generators is proven in Section 2.3. Together these formulas yield the proof of Theorem 1.16, completed in Section 2.4.

2.1 A helpful lemma via Pick's theorem

First, we fix some notation. For any convex generator Λ , let P_Λ be the region bounded by Λ and the x - and y -axes. We define $\mathbb{A}(\Lambda)$ to be the *area of* P_Λ , and $\mathbb{A}_\nu(\Lambda)$ to be the *area of the portion of* P_Λ *lying underneath* ν .

Definition 2.1. For any convex generator Λ , we define

$$b(\Lambda) = x(\Lambda) + y(\Lambda) + m(\Lambda).$$

Recall that the formal product 1 is the path Λ with no edges which starts and ends at $(0,0)$. Note that $b(\Lambda)$ computes the lattice points on the boundary of any $\Lambda \neq 1$ if and only if Λ does not lie entirely on one axis.

Remark 2.2. The operator b is additive under products of convex generators. In other words, for any convex generators Λ and Γ , we have

$$\begin{aligned} b(\Lambda \cdot \Gamma) &= x(\Lambda \cdot \Gamma) + y(\Lambda \cdot \Gamma) + m(\Lambda \cdot \Gamma) \\ &= x(\Lambda) + x(\Gamma) + y(\Lambda) + y(\Gamma) + m(\Lambda) + m(\Gamma) \\ &= (x(\Lambda) + y(\Lambda) + m(\Lambda)) + (x(\Gamma) + y(\Gamma) + m(\Gamma)) \\ &= b(\Lambda) + b(\Gamma). \end{aligned}$$

Using the above notation, we can now prove a useful formula for the index of a convex generator.

Lemma 2.3. *Let Λ be any convex generator. Then,*

$$I(\Lambda) = 2\mathbb{A}(\Lambda) + b(\Lambda) - h(\Lambda). \quad (2)$$

Proof. First, suppose that Λ lies entirely on one axis. If $\Lambda = e_{1,0}^x$ for some $x \geq 0$, we have

$$I(\Lambda) = 2x = 2 \cdot 0 + 2x - 0 = 2\mathbb{A}(\Lambda) + b(\Lambda) - h(\Lambda).$$

The case where $\Lambda = e_{0,1}^y$ for some $y \geq 0$ is analogous.

Next, suppose that Λ does not lie entirely on one axis. Let P be the region bounded by Λ and the x - and y -axes. Then, Pick's Theorem states that

$$\mathbb{A}(\Lambda) = i(P) + \frac{b(P)}{2} - 1,$$

where $i(P)$ is the number of lattice points in the interior of P and $b(P)$ is the number of lattice points on the boundary of P . Rearranging and noting that $L(\Lambda) = i(P) + b(P)$, we obtain

$$L(\Lambda) = i(P) + b(P) = \mathbb{A}(\Lambda) + \frac{b(P)}{2} + 1 = \mathbb{A}(\Lambda) + \frac{b(\Lambda)}{2} + 1,$$

where the last equality follows from the fact that Λ does not lie entirely on one axis. We can then use this expression for $L(\Lambda)$ to compute $I(\Lambda)$:

$$I(\Lambda) = 2(L(\Lambda) - 1) - h(\Lambda) = 2\mathbb{A}(\Lambda) + b(\Lambda) - h(\Lambda). \quad (3)$$

□

2.2 Repeated factors in the Hutchings criterion

Our first result in this section is the following proposition.

Proposition 2.4. *Let X_Ω and $X_{\Omega'}$ be convex toric domains, and let Λ and Λ' be nontrivial convex generators with no edges labelled ‘h.’ Suppose that $\Lambda \leq_{\Omega, \Omega'} \Lambda'$ and $I(\Lambda \cdot \Lambda) = I(\Lambda' \cdot \Lambda')$. Then,*

$$8\mathbb{A}(\Lambda') \leq (b(\Lambda') - 1)^2.$$

Moreover, Λ must be of the form $e_{x,y}$, where $x, y \in \mathbb{Z}_{>0}$ are coprime and satisfy

$$xy = 2\mathbb{A}(\Lambda')$$

and

$$x + y = b(\Lambda') - 1.$$

Equivalently, x and y must be nonnegative coprime integers such that

$$\{x, y\} = \left\{ \frac{b(\Lambda') - 1 \pm \sqrt{(b(\Lambda') - 1)^2 - 8\mathbb{A}(\Lambda')}}{2} \right\}. \quad (4)$$

Proof. We will make repeated use of Lemma 2.3, aka (2). Using (2) along with the additivity of b , we get

$$\begin{aligned} I(\Lambda \cdot \Lambda) &= 2\mathbb{A}(\Lambda \cdot \Lambda) + b(\Lambda \cdot \Lambda) - h(\Lambda \cdot \Lambda) \\ &= 2\mathbb{A}(\Lambda \cdot \Lambda) + 2b(\Lambda) \end{aligned} \quad (5)$$

Recall that P_Λ denotes the region bounded by Λ and the x - and y -axes. Then, the region bounded by $\Lambda \cdot \Lambda$ and the x - and y -axes is P_Λ dilated by a factor of 2, which has 4 times the area of P_Λ , e.g.

$$\mathbb{A}(\Lambda \cdot \Lambda) = 4\mathbb{A}(\Lambda). \quad (6)$$

Substituting (6) into the above equation (5) and using (2) again yields,

$$\begin{aligned} I(\Lambda \cdot \Lambda) &= 8\mathbb{A}(\Lambda) + 2b(\Lambda) \\ &= 4\mathbb{A}(\Lambda) + 2(2\mathbb{A}(\Lambda) + b(\Lambda)) \\ &= 4\mathbb{A}(\Lambda) + 2I(\Lambda), \end{aligned}$$

Likewise for Λ' we obtain,

$$I(\Lambda' \cdot \Lambda') = 4\mathbb{A}(\Lambda') + 2I(\Lambda').$$

We assumed that $I(\Lambda \cdot \Lambda) = I(\Lambda' \cdot \Lambda')$, thus,

$$4\mathbb{A}(\Lambda) + 2I(\Lambda) = 4\mathbb{A}(\Lambda') + 2I(\Lambda').$$

Since $I(\Lambda) = I(\Lambda')$, we have

$$\mathbb{A}(\Lambda) = \mathbb{A}(\Lambda'). \quad (7)$$

Now, because of (2), we have

$$I(\Lambda) = 2\mathbb{A}(\Lambda) + b(\Lambda) = I(\Lambda') = 2\mathbb{A}(\Lambda') + b(\Lambda').$$

Combining this equation with (7) gives

$$b(\Lambda) = x(\Lambda) + y(\Lambda) + m(\Lambda) = b(\Lambda'). \quad (8)$$

On the other hand, the fact that $\Lambda \leq_{\Omega, \Omega'} \Lambda'$ implies that

$$x(\Lambda) + y(\Lambda) \geq b(\Lambda') - 1. \quad (9)$$

Since $m(\Lambda) > 0$, the only way that (8) and (9) can simultaneously be true is if (9) is an equality and we have $m(\Lambda) = 1$. So, Λ must have the form $e_{x,y}$, where $\gcd(x, y) = 1$. This allows us to compute properties of Λ explicitly, so that (9) becomes

$$x(\Lambda) + y(\Lambda) = x + y = b(\Lambda') - 1, \quad (10)$$

and (7) becomes

$$\mathbb{A}(\Lambda) = \frac{xy}{2} = \mathbb{A}(\Lambda'),$$

or equivalently

$$xy = 2\mathbb{A}(\Lambda'). \quad (11)$$

We can now use (10) and (11) to solve for x and y . Multiplying (10) by x and substituting in (11) gives

$$x^2 + xy = x^2 + 2\mathbb{A}(\Lambda') = x(b(\Lambda') - 1).$$

The quadratic formula then implies that

$$x = \frac{b(\Lambda') - 1 \pm \sqrt{(b(\Lambda') - 1)^2 - 8\mathbb{A}(\Lambda')}}{2}.$$

For either of these two choices of x , we can use (11) to determine what y must be. Thus, there are only two possible choices of x and y . On the other hand, since (10) and (11) are symmetric in x and y , $e_{y,x}$ must also be a valid choice for Λ . This implies that for either possible choice of x , y must be equal to the other possible choice of x . It then follows that both choices for x and y are specified by (4). Finally, we note that since x and y are real, the square roots in (4) must be real, i.e.

$$8\mathbb{A}(\Lambda') \leq (b(\Lambda') - 1)^2.$$

□

Remark 2.5. It is helpful to understand when the converse to the Proposition 2.4 holds. Given any convex toric domains X_Ω and $X_{\Omega'}$ and any convex generator Λ' , suppose x and y satisfy,

$$\{x, y\} = \left\{ \frac{b(\Lambda') - 1 \pm \sqrt{(b(\Lambda') - 1)^2 - 8\mathbb{A}(\Lambda')}}{2} \right\}.$$

One can show that if x and y are nonnegative, coprime integers such that $A_\Omega(e_{x,y}) \leq A_{\Omega'}(\Lambda')$, then we have $e_{x,y} \leq_{\Omega, \Omega'} \Lambda'$ and $I(e_{x,y} \cdot e_{x,y}) = I(\Lambda' \cdot \Lambda')$. Thus, to obstruct the existence of Λ as in the statement of the proposition, we must show that the above formulas for x and y either will not yield nonnegative, coprime integers or will imply that $A_\Omega(e_{x,y}) > A_{\Omega'}(\Lambda')$.

Remark 2.6. The significance of the above proposition to the Hutchings criterion is as follows. Suppose we have $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$ satisfying the conditions of Theorem 1.14. If there exists some $i \neq j$ such that $\Lambda_i = \Lambda_j$ and $\Lambda'_i = \Lambda'_j$, then conditions (i) and (iii) of Theorem 1.14 imply that the assumptions of Proposition 1.18 are satisfied. So, the proposition tells us that $\Lambda_i = e_{x,y}$, where x and y can be determined from properties of Λ'_i using (1).

Other implications of Proposition 2.4 with regard to the Hutchings criterion were discussed in Remark 1.19.

2.3 The index of the product of two convex generators

The remainder of this section is devoted to proving Theorem 1.16. To prove this theorem, we need to understand the index of the product of two convex generators. We will demonstrate a formula for such an index shortly. The formula itself is somewhat complicated, so we first provide an example to elucidate the intuition behind it.

First, we need some additional notation. For any convex generator Λ and any edge ν of Λ , we write ν_x and ν_y for the x - and y -coordinates of the displacement vector of ν . We also define the *slope* of ν to be,

$$\mu(\nu) = \frac{\nu_y}{\nu_x}.$$

Example. Let $\Lambda = e_{1,0}^3 e_{2,1} e_{1,3}$, and let $\Gamma = e_{2,1} e_{0,1}^2$. Using (2) along with the additivity of b and h , we have,

$$\begin{aligned} I(\Lambda \cdot \Gamma) &= 2\mathbb{A}(\Lambda \cdot \Gamma) + b(\Lambda \cdot \Gamma) - h(\Lambda \cdot \Gamma) \\ &= 2\mathbb{A}(\Lambda \cdot \Gamma) + b(\Lambda) + b(\Gamma) - h(\Lambda) - h(\Gamma). \end{aligned} \tag{12}$$

We can compute $\mathbb{A}(\Lambda \cdot \Gamma)$ by summing the area under each of the edges of $\Lambda \cdot \Gamma = e_{1,0}^3 e_{2,1}^2 e_{1,3} e_{0,1}^2$.

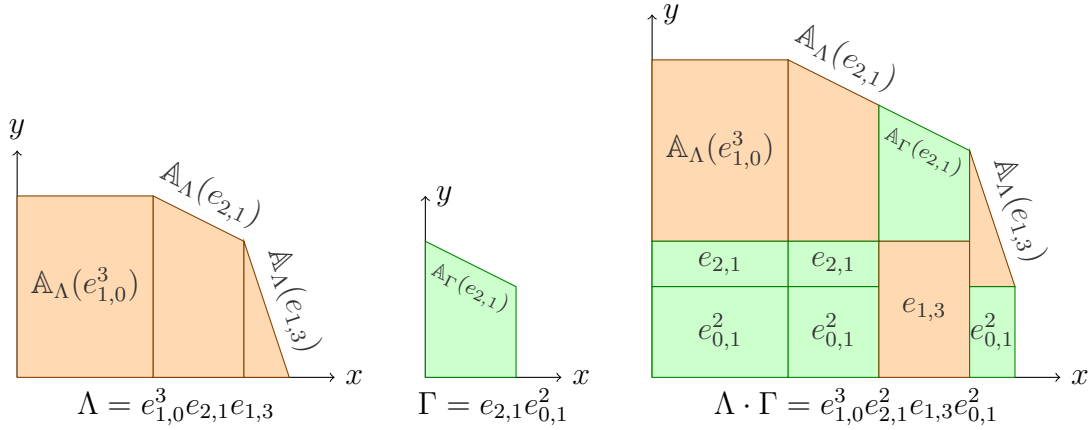


Figure 1: The graph on the right shows $\Lambda \cdot \Gamma$ broken up into pieces of area from Λ and Γ along with rectangles added by taking the product. Rectangles that were added by taking the product are labelled with the edge that necessitated that rectangle. The graphs on the left and center show Λ and Γ for comparison.

For any edge ν of Λ , the region underneath ν in $\Lambda \cdot \Gamma$ will be essentially the same shape as the region under ν in Λ , except that ν may be higher up (i.e. its endpoints may have larger y -coordinates) in the product $\Lambda \cdot \Gamma$. To see this, notice that the y -coordinate of the lower right endpoint of ν in Λ is

$$y_\Lambda = \sum_{\substack{\sigma \in \text{Edges}(\Lambda) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y,$$

while the y -coordinate of the lower right endpoint of ν in $\Lambda \cdot \Gamma$ is

$$y_{\Lambda \cdot \Gamma} = \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y.$$

Thus, every edge σ of Γ that is steeper than ν will contribute a term of σ_y to $y_{\Lambda \cdot \Gamma}$ which is not in y_Λ , so that the edge ν in $\Lambda \cdot \Gamma$ will be translated upwards by σ_y relative to the position of ν in Λ . This translation is equivalent to taking the region beneath ν in Λ and adding a rectangle to the bottom of it. So, $A_{\Lambda \cdot \Gamma}(\nu)$ will be equal to $A_\Lambda(\nu)$ plus the area of several rectangle added beneath ν . Thinking of area in this way allows us to break up the area under each edge in $\Lambda \cdot \Gamma$ into individual contributions from different edges, as shown in Figure 1.

One important feature of this figure is how we split up the area under the edge $e_{2,1}^2$ in $\Lambda \cdot \Gamma$. Because both Λ and Γ have an edge of slope $-1/2$, we treat these as separate and compute areas underneath them individually, even though they combine to form one edge in $\Lambda \cdot \Gamma$. This is important because whichever copy of $e_{2,1}$ is on the left (in the figure we've shown it as the one from Λ , but it would not have affected the

answer if we'd put the one from Γ on the left instead) has one rectangle underneath it contributed by the other copy of $e_{2,1}$.

We can now compute $\mathbb{A}(\Lambda \cdot \Gamma)$ by summing up the area contributions of each region of $\Lambda \cdot \Gamma$ shown in Figure 1. Let R be the sum of the areas of all the rectangles added by taking the product as described above (that is, all the rectangles underneath $\Lambda \cdot \Gamma$ in the figure except the one labelled $\mathbb{A}_\Lambda(e_{1,0}^3)$). Then,

$$\begin{aligned}\mathbb{A}(\Lambda \cdot \Gamma) &= \mathbb{A}_\Lambda(e_{1,0}^3) + \mathbb{A}_\Lambda(e_{2,1}) + \mathbb{A}_\Gamma(e_{2,1}) + \mathbb{A}_\Lambda(e_{1,3}) + R \\ &= \mathbb{A}(\Lambda) + \mathbb{A}(\Gamma) + R.\end{aligned}$$

Plugging back into (12) and applying (2) then gives

$$\begin{aligned}I(\Lambda \cdot \Gamma) &= 2(\mathbb{A}(\Lambda) + \mathbb{A}(\Gamma) + R) + b(\Lambda) + b(\Gamma) - h(\Lambda) - h(\Gamma) \\ &= (2\mathbb{A}(\Lambda) + b(\Lambda) - h(\Lambda)) + (2\mathbb{A}(\Gamma) + b(\Gamma) - h(\Gamma)) + 2R \\ &= I(\Lambda) + I(\Gamma) + 2R.\end{aligned}\tag{13}$$

Equation (13) is precisely the sort of expression we want for the index of the product of two convex generators. By generalizing the above arguments as follows, we obtain an formula for the product of two arbitrary generators with no hyperbolic orbit in common, with an explicit expression for R .

Proposition 2.7. *Let Λ and Γ be any two convex generators that have no hyperbolic orbit in common. Then,*

$$I(\Lambda \cdot \Gamma) = I(\Lambda) + I(\Gamma) + 2 \sum_{\nu \in \text{Edges}(\Lambda)} \sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Gamma)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y.$$

Proof. As in (12), we use (2) along with the additivity of b and h to get,

$$I(\Lambda \cdot \Gamma) = 2\mathbb{A}(\Lambda \cdot \Gamma) + b(\Lambda) + b(\Gamma) - h(\Lambda) - h(\Gamma).\tag{14}$$

Next, we compute $\mathbb{A}(\Lambda \cdot \Gamma)$ by summing the area underneath each edge. First, consider any edge ν of Λ whose slope is not equal to that of any edge of Γ . Let y_1 and y_2 be the y -coordinates of the upper left and lower right endpoints (respectively) of ν in $\Lambda \cdot \Gamma$. Then, the region underneath ν is either a trapezoid or a triangle; in either case, its area is

$$\mathbb{A}_{\Lambda \cdot \Gamma}(\nu) = \frac{\nu_x(y_1 + y_2)}{2} = \frac{\nu_x(2y_2 + \nu_y)}{2},\tag{15}$$

where the last equality follows from the fact that $y_1 = y_2 + \nu_y$. Notice that y_2 is equal to the sum of the y -coordinates of the edges to the right of ν in $\Lambda \cdot \Gamma$. In equations:

$$y_2 = \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y.$$

Plugging this into our equation for $\mathbb{A}_{\Lambda \cdot \Gamma}(\nu)$ gives

$$\mathbb{A}_{\Lambda \cdot \Gamma}(\nu) = \frac{\nu_x}{2} \left(\nu_y + 2 \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right). \quad (16)$$

Now, all of the above arguments also go through if we consider edges ν in Λ , so we get an identical formula to (16) for $\mathbb{A}_{\Lambda}(\nu)$, except with the sum taken only over the edges of Λ . Subtracting this formula from (16) gives,

$$\begin{aligned} \mathbb{A}_{\Lambda \cdot \Gamma}(\nu) - \mathbb{A}_{\Lambda}(\nu) &= \frac{\nu_x}{2} \left(\nu_y + 2 \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y - \nu_y - 2 \sum_{\substack{\sigma \in \text{Edges}(\Lambda) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right) \\ &= \nu_x \left(\sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right). \end{aligned}$$

Rearranging, we obtain,

$$\mathbb{A}_{\Lambda \cdot \Gamma}(\nu) = \mathbb{A}_{\Lambda}(\nu) + \nu_x \left(\sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right). \quad (17)$$

The same arguments apply to edges ν in Γ , provided we swap Λ with Γ in our formulas. Thus, (17) also holds for any edge ν of Γ whose slope is not equal to that of any edge of Λ .

Next, we consider any edge ν of Λ such that there exists some edge ν' of Γ with $\mu(\nu) = \mu(\nu')$. The edges ν and ν' combine to form a single edge in the product $\Lambda \cdot \Gamma$; however, we will compute the area under the portion of the edge contributed by ν separately from the area under the portion of the edge contributed by ν' . To this end, we will speak of “the edge ν (respectively ν') in $\Lambda \cdot \Gamma$ ” when we mean the portion of the joint edge in $\Lambda \cdot \Gamma$ contributed by ν (respectively ν').

We will also need to choose which of ν and ν' comes at the beginning (i.e. on the left) of the joint edge in $\Lambda \cdot \Gamma$. Note that this choice cannot affect our computation: we can swap the placement of the two portions of the joint edge without changing the convex generator $\Lambda \cdot \Gamma$ at all. So, without loss of generality, we will consider ν to lie to the left of ν' in $\Lambda \cdot \Gamma$.

We will compute $\mathbb{A}_{\Lambda \cdot \Gamma}(\nu)$ much as we did in the case where Γ has no edge of slope $\mu(\nu)$. Let y_1 and y_2 be the upper left and lower right endpoints (respectively) of the

edge ν in $\Lambda \cdot \Gamma$. Then, because ν' is to the right of ν in $\Lambda \cdot \Gamma$, we have,

$$y_2 = \nu'_y + \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y.$$

Using the same area formula (15) as before, we then obtain,

$$\mathbb{A}_{\Lambda \cdot \Gamma}(\nu) = \frac{\nu_x}{2} \left(\nu_y + 2 \left(\nu'_y + \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right) \right).$$

However, a formula analogous to (16) is still applicable for $A_\Lambda(\nu)$, so subtracting that from the above equation gives

$$\begin{aligned} \mathbb{A}_{\Lambda \cdot \Gamma}(\nu) - \mathbb{A}_\Lambda(\nu) &= \frac{\nu_x}{2} \left(\nu_y + 2\nu'_y + 2 \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y - \nu_y - 2 \sum_{\substack{\sigma \in \text{Edges}(\Lambda) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right) \\ &= \nu_x \left(\nu'_y + \sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right) \\ &= \nu_x \sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) \leq \mu(\nu)}} \sigma_y. \end{aligned}$$

Rearranging, we obtain,

$$\mathbb{A}_{\Lambda \cdot \Gamma}(\nu) = \mathbb{A}_\Lambda(\nu) + \nu_x \sum_{\substack{\sigma \in \text{Edges}(\Lambda \cdot \Gamma) \\ \mu(\sigma) \leq \mu(\nu)}} \sigma_y. \quad (18)$$

In the case where Γ has no edge of slope $\mu(\nu)$, (18) is equivalent to (17). So, we may actually use (18) to compute the area under any edge in $\Lambda \cdot \Gamma$ coming from Λ .

The only difference between our derivation of (17) and (18) is that in the latter case, we added an extra term of ν'_y to y_2 . However, because ν' is to the right of ν in $\Lambda \cdot \Gamma$, we do not need to add any such extra term when computing the y -coordinate of the lower right endpoint of ν' . As a result, the computation for $A_{\Lambda \cdot \Gamma}(\nu')$ goes through exactly as it would in the case of an edge that does not share a slope with any edge of Λ , which means we can use (17) to compute the area under ν' in $\Lambda \cdot \Gamma$. Thus, we can use (17) for every edge in $\Lambda \cdot \Gamma$ coming from Γ .

With this, we are ready to compute $\mathbb{A}(\Lambda \cdot \Gamma)$. We just have to sum the righthand side of (18) for all the edges of Λ and sum the righthand side of (17) for all the edges

of Γ . This yields,

$$\begin{aligned}\mathbb{A}(\Lambda \cdot \Gamma) &= \sum_{\nu \in \text{Edges}(\Lambda)} \left(\mathbb{A}_\Lambda(\nu) + \nu_x \sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) \leq \mu(\nu)}} \sigma_y \right) + \sum_{\nu \in \text{Edges}(\Gamma)} \left(\mathbb{A}_\Gamma(\nu) + \nu_x \sum_{\substack{\sigma \in \text{Edges}(\Lambda) \\ \mu(\sigma) < \mu(\nu)}} \sigma_y \right) \\ &= \mathbb{A}(\Lambda) + \mathbb{A}(\Gamma) + \sum_{\nu \in \text{Edges}(\Lambda)} \sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + \sum_{\nu \in \text{Edges}(\Gamma)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y.\end{aligned}$$

Finally, we plug this area into (14):

$$\begin{aligned}I(\Lambda \cdot \Gamma) &= 2 \left(\mathbb{A}(\Lambda) + \mathbb{A}(\Gamma) + \sum_{\nu \in \text{Edges}(\Lambda)} \sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + \sum_{\nu \in \text{Edges}(\Gamma)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y \right) \\ &\quad + b(\Lambda) + b(\Gamma) - h(\Lambda) - h(\Gamma).\end{aligned}$$

Rearranging and applying (2) then yields the desired result. \square

2.4 Proof of Theorem 1.16

With Proposition 2.7 in hand, we are now ready to prove Theorem 1.16.

Proof of Theorem 1.16. First, notice that for any $i \neq j$, our assumptions along with Proposition 2.7 imply that,

$$\begin{aligned}I(\Lambda'_i \Lambda'_j) &= I(\Lambda'_i) + I(\Lambda'_j) + 2 \sum_{\nu \in \text{Edges}(\Lambda'_i)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_j) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda'_j)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_i) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y \\ &= I(\Lambda_i) + I(\Lambda_j) + 2 \sum_{\nu \in \text{Edges}(\Lambda'_i)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_j) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda'_j)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_i) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y.\end{aligned}\tag{19}$$

On the other hand, by much the same argument,

$$\begin{aligned}I(\Lambda'_i \Lambda'_j) &= I(\Lambda_i \Lambda_j) \\ &= I(\Lambda_i) + I(\Lambda_j) + 2 \sum_{\nu \in \text{Edges}(\Lambda_i)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda_j) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda_j)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda_i) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y.\end{aligned}$$

Equating this to (19) and cancelling the $I(\Lambda_i)$'s and $I(\Lambda_j)$'s yields

$$\begin{aligned}
2 \sum_{\nu \in \text{Edges}(\Lambda'_i)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_j) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda'_j)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_i) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y = \\
2 \sum_{\nu \in \text{Edges}(\Lambda_i)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda_j) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda_j)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda_i) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y. \quad (20)
\end{aligned}$$

This equality will be useful to us shortly.

We will proceed by induction on the size of the subset S . In the base cases, where $|S| = 1$ or $|S| = 2$, the desired statement is given by assumption. So, suppose that for some $3 \leq k \leq n$, the desired statement holds for any set $S \subseteq \{1, 2, \dots, n\}$ such that $|S| = k - 1$. Let $S \subset \{1, \dots, n\}$ such that $|S| = k$, and fix some $i_0 \in S$. Then, we can write,

$$\prod_{i \in S} \Lambda'_i = \Gamma' \cdot \Lambda'_{i_0} \quad \text{and} \quad \prod_{i \in S} \Lambda_i = \Gamma \cdot \Lambda_{i_0},$$

where,

$$\Gamma' = \prod_{i \in S \setminus \{i_0\}} \Lambda'_i \quad \text{and} \quad \Gamma = \prod_{i \in S \setminus \{i_0\}} \Lambda_i.$$

We wish to show that $I(\Gamma' \cdot \Lambda'_{i_0}) = I(\Gamma \cdot \Lambda_{i_0})$. Proposition 2.7 implies that,

$$I(\Gamma' \cdot \Lambda'_{i_0}) = I(\Gamma') + I(\Lambda'_{i_0}) + 2 \sum_{\nu \in \text{Edges}(\Gamma')} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_{i_0}) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda'_{i_0})} \sum_{\substack{\sigma \in \text{Edges}(\Gamma') \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y.$$

By assumption, we have $I(\Lambda'_{i_0}) = I(\Lambda_{i_0})$, and by the induction hypothesis, we have $I(\Gamma') = I(\Gamma)$.

Moreover, we can replace $\text{Edges}(\Gamma')$ with $\bigsqcup_{i \neq i_0} \text{Edges}(\Lambda'_i)$ in the above equation without changing it. This replacement amounts to splitting into pieces all of the edges in Γ' that are combinations of multiple edges of the same slope coming from different Λ'_i , and such a splitting does not affect the values of the sums we are dealing with. Putting all of this together, we have,

$$I(\Gamma' \cdot \Lambda'_{i_0}) = I(\Gamma) + I(\Lambda_{i_0}) + \sum_{i \neq i_0} \left(2 \sum_{\nu \in \text{Edges}(\Lambda'_i)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_{i_0}) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda'_{i_0})} \sum_{\substack{\sigma \in \text{Edges}(\Lambda'_i) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y \right).$$

We can then apply (20) to get,

$$I(\Gamma' \cdot \Lambda'_{i_0}) = I(\Gamma) + I(\Lambda_{i_0}) + \sum_{i \neq i_0} \left(2 \sum_{\nu \in \text{Edges}(\Lambda_i)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda_{i_0}) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda_{i_0})} \sum_{\substack{\sigma \in \text{Edges}(\Lambda_i) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y \right).$$

Now, the same argument that allowed us to substitute $\bigsqcup_{i \neq i_0} \text{Edges}(\Lambda'_i)$ for $\text{Edges}(\Gamma')$ allows us to get rid of the outer sum in the above equation and take the inner sums over $\text{Edges}(\Gamma)$ instead of $\text{Edges}(\Lambda_i)$. We obtain,

$$I(\Gamma' \cdot \Lambda'_{i_0}) = I(\Gamma) + I(\Lambda_{i_0}) + 2 \sum_{\nu \in \text{Edges}(\Gamma)} \sum_{\substack{\sigma \in \text{Edges}(\Lambda_{i_0}) \\ \mu(\sigma) \leq \mu(\nu)}} \nu_x \sigma_y + 2 \sum_{\nu \in \text{Edges}(\Lambda_{i_0})} \sum_{\substack{\sigma \in \text{Edges}(\Gamma) \\ \mu(\sigma) < \mu(\nu)}} \nu_x \sigma_y.$$

However, Proposition 2.7 tells us that the righthand side of this equation is equal to $I(\Gamma \cdot \Lambda_{i_0})$. So, we have proven that $I(\Gamma' \cdot \Lambda'_{i_0}) = I(\Gamma \cdot \Lambda_{i_0})$, as desired. \square

3 On symplectic embeddings of a polydisk into a ball

Our main goal in this section is to prove Theorem 1.2, that for $2 \leq a < \frac{\sqrt{7}-1}{\sqrt{7}-2}$ and if $P(a, 1)$ symplectically embeds into $B(c)$ then $c \geq 2 + a/2$.

The plan of attack is to assume the statement of the theorem is false and apply the modified Hutchings criterion, Theorem 1.14, to the generator $\Lambda' = e_{1,1}^d$ for a suitable choice of d . By [8, Lemma 2.1] this is a minimal generator for $B(c)$. This gives us an integer n , a convex generator Λ , and factorizations $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$. To obtain a contradiction, we show that no choice of the Λ'_i and Λ_i is possible. We do so in three steps.

1. We prove that for sufficiently large m , there is no convex generator Λ such that $\Lambda \leq_{P(a,1), B(c)} e_{1,1}^m$. If we choose d to be very large, this will imply that we cannot have $n = 1$. This step is the content of Proposition 3.2, which is proved in Section 3.1.
2. We use Proposition 2.4 to show that there cannot exist any $i \neq j$ such that $\Lambda'_i = \Lambda'_j$ and $\Lambda_i = \Lambda_j$. In conjunction with Step 1, this will imply that the set of all possible values of n is bounded. This step is the content of Proposition 3.3, which is proved in Section 3.2.
3. Using Steps 1 and 2, we show that there is a maximum possible index of the product $\prod_{i=1}^n \Lambda'_i$ which does not depend on d . On the other hand, this product must be equal to $\Lambda' = e_{1,1}^d$. Because of Step 1, we will be able to pick d to be arbitrarily large, which will make the index of Λ' arbitrarily large, resulting in a contradiction. This step is contained in the proof of Theorem 1.2, which is given in Section 3.3.

After we have proven Theorem 1.2, we will discuss whether it is possible for an application of the Hutchings criterion to extend the results of the theorem. This discussion is the content of Sections 3.4 and 3.5.

3.1 Elimination of sufficiently large convex generators

We first prove some useful inequalities on the x and y endpoints of certain convex generators.

Lemma 3.1. *Let $a > 1$ and $c < 2 + a/2$, and suppose d and Λ are such that $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$. Then,*

$$x(\Lambda) < \left(2 + \frac{a}{2}\right) d - ay(\Lambda), \quad (21)$$

and

$$y(\Lambda) < \frac{d(a-2) + 2}{2(a-1)}. \quad (22)$$

Proof. Our assumptions tell us,

$$A_{P(a,1)}(\Lambda) = x(\Lambda) + ay(\Lambda) \leq A_{B(c)}(e_{1,1}^d) = cd < \left(2 + \frac{a}{2}\right) d \quad (23)$$

and,

$$x(\Lambda) + y(\Lambda) \geq x(e_{1,1}^d) + y(e_{1,1}^d) + m(e_{1,1}^d) - 1 = 3d - 1. \quad (24)$$

We can immediately solve for $x(\Lambda)$ in (23),

$$x(\Lambda) < \left(2 + \frac{a}{2}\right) d - ay(\Lambda).$$

Alternately, combining (23) and (24) gives,

$$3d - 1 + (a-1)y(\Lambda) \leq x(\Lambda) + ay(\Lambda) < \left(2 + \frac{a}{2}\right) d.$$

Solving for $y(\Lambda)$ shows,

$$y(\Lambda) < \frac{d\left(\frac{a}{2} - 1\right) + 1}{a-1} = \frac{d(a-2) + 2}{2(a-1)}.$$

□

We now use the above lemma to eliminate sufficiently large convex generators from consideration in the proof of Theorem 1.2.

Proposition 3.2. *Let*

$$2 \leq a < \frac{\sqrt{7} - 1}{\sqrt{7} - 2},$$

and suppose that $c < 2 + a/2$. Then, there exists some $d_a \geq 1$ such that, for any $d > d_a$ and any convex generator Λ , $\Lambda \not\leq_{P(a,1),B(c)} e_{1,1}^d$.

Proof. Fix some d , and suppose there exists $\Lambda \leq e_{1,1}^d$. Let $x = x(\Lambda)$ and $y = y(\Lambda)$. Because Λ is convex, it lies inside the rectangle $[0, x] \times [0, y]$. Thus, the maximum possible value of $L(\Lambda)$ occurs when Λ contains all the lattice points inside this rectangle, and the largest $I(\Lambda)$ could be is when Λ contains all these lattice points and has no edges labelled ‘ h .’ Noting also that $I(\Lambda) = I(e_{1,1}^d) = d(d+3)$, we see that

$$2((x+1)(y+1) - 1) = 2(x+1)(y+1) - 2 \geq I(\Lambda) = d(d+3),$$

or equivalently,

$$0 \geq d(d+3) + 2 - 2(x+1)(y+1).$$

The substitution of (21) into this yields,

$$\begin{aligned} 0 &> d(d+3) + 2 - 2\left(\left(2 + \frac{a}{2}\right)d - ay + 1\right)(y+1); \\ 0 &> d(d+3) + 2 - (4+a)dy + 2ay^2 - 2y - (4+a)d + 2ay - 2. \end{aligned}$$

Collecting powers of y yields,

$$0 > 2ay^2 - y((4+a)d + 2 - 2a) + d(d+3) - (4+a)d. \quad (25)$$

We now wish to substitute (22) into (25), while still maintaining a valid inequality. This is permissible provided the right hand side of (25) is nonincreasing with respect to increasing y . Notice that the derivative of the right hand side of (25) with respect to y is

$$4ay - (4+a)d - 2 + 2a.$$

Substituting (22) into this expression gives us,

$$\begin{aligned} 4ay - (4+a)d - 2 + 2a &< \frac{2ad(a-2) + 4a}{a-1} - (4+a)d - 2 + 2a; \\ &< \frac{d(a^2 - 7a + 4) + 2(a^2 + 1)}{a-1}. \end{aligned} \quad (26)$$

Now, $a^2 - 7a + 4$ has roots at $a = \frac{7 \pm \sqrt{33}}{2} \approx 0.628, 6.372$. Since a is in between these two roots, we have $a^2 - 7a + 4 < 0$. So, the expression in (26) will be negative for all d above some sufficiently large value d_1 . In this case, we can substitute (22) into the lefthand side of (25) obtaining,

$$0 > \frac{a(d(a-2) + 2)^2}{2(a-1)^2} - \frac{d(a-2) + 2}{2(a-1)}[(4+a)d - 2(a-1)] + d(d+3) - (4+a)d.$$

Multiplying by $2(a-1)$ and collecting powers of d yields,

$$0 > d^2 \left[\frac{a(a-2)^2}{a-1} - (a-2)(4+a) + 2(a-1) \right] + \\ d \left[\frac{4a(a-2)}{a-1} - 2(4+a) + 2(a-2)(a-1) + 6(a-1) - 2(4+a)(a-1) \right] + \\ \left[\frac{4a}{a-1} + 4(a-1) \right].$$

Simplifying this gives us,

$$0 > \left(\frac{-3a^2 + 10a - 6}{a-1} \right) d^2 + \left(\frac{-4a^2 - 2a + 2}{a-1} \right) d + \frac{4(a^2 - a + 1)}{a-1}.$$

Multiplying by $(a-1)$ produces,

$$0 > (-3a^2 + 10a - 6)d^2 - 2(2a^2 + a - 1)d + 4(a^2 - a + 1). \quad (27)$$

Now, the coefficient of d^2 in (27) is negative for sufficiently large a and has roots at $a = \frac{5 \pm \sqrt{7}}{3} \approx 0.7848, 2.5486$. Because our value of a is between these two roots, we can conclude that the coefficient of d^2 is positive. Thus, if d is larger than some sufficiently large value d_2 , the righthand side of (27) will be positive, a contradiction.

We have shown that if $d > d_1$ and $d > d_2$, then the existence of Λ results in a contradiction. Since d_1 and d_2 depend only on a by construction, setting $d_a = \max\{d_1, d_2\}$ now yields the desired statement. \square

3.2 Elimination of repeated factors of convex generators

Proposition 3.3. *Let $2 \leq a \leq 3$, $c < 2 + a/2$, and $d \geq 1$. Then, for any convex generator Λ , either $\Lambda \not\leq_{P(a,1),B(c)} e_{1,1}^d$, or $I(\Lambda \cdot \Lambda) \neq I(e_{1,1}^{2d})$.*

Proof. To obtain a contradiction, suppose that there exists some Λ such that $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$ and $I(\Lambda \cdot \Lambda) = I(e_{1,1}^{2d})$. Then, we can apply Proposition 2.4 with $\Lambda' = e_{1,1}^d$. Noting that $A(\Lambda') = d^2/2$ and $b(\Lambda') = 3d$, we get $\Lambda = e_{x,y}$, where,

$$x = \frac{3d - 1 \pm \sqrt{5d^2 - 6d + 1}}{2} \quad (28)$$

and,

$$y = \frac{d^2}{x}. \quad (29)$$

On the other hand, $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$ implies,

$$A_{P(a,1)}(\Lambda) = x(\Lambda) + ay(\Lambda) = x + ay \leq A_{B(c)}(e_{1,1}^d) = cd < \left(2 + \frac{a}{2}\right) d.$$

Substituting in our expression (29) for y and multiplying by x gives,

$$x^2 + ad^2 < \left(2 + \frac{a}{2}\right)xd.$$

We then substitute in our expression (28) for x and multiply by 4 to get

$$(3d-1)^2 \pm (6d-2)\sqrt{5d^2-6d+1} + 5d^2 - 6d + 1 + 4ad^2 < (4+a)d \left(3d-1 \pm \sqrt{5d^2-6d+1}\right),$$

or equivalently,

$$(2+a)d^2 + (a-8)d + 2 \pm (2d-2-ad)\sqrt{5d^2-6d+1} < 0.$$

The lefthand side of this equation can be factored:

$$\left(-d-1 \pm \sqrt{5d^2-6d+1}\right) \left((3-a)d-1 \pm \sqrt{1-6d+5d^2}\right) < 0. \quad (30)$$

The zeros of the left factor (if they exist) occur when,

$$(-d-1)^2 = 5d^2 - 6d + 1,$$

i.e. when $d = 0$ or $d = 2$. Likewise, the zeros of the right factor (if they exist) occur when,

$$((3-a)d-1)^2 = 5d^2 - 6d + 1,$$

or equivalently when,

$$d((-a^2 + 6a - 4)d - 2a) = 0.$$

This equation holds when $d = 0$ and when $d = 2a/(-a^2 + 6a - 4)$. Note that for all $2 \leq a \leq 3$, we have $1 \leq 2a/(-a^2 + 6a - 4) < 2$.

Suppose the sign of the square roots in (30) is positive. Then, both factors in (30) go to ∞ as $d \rightarrow \infty$, and both possible zeros of both factors are actually zeros of these factors. If $d \geq 2$, then d is at least as large as all of the zeros of the lefthand side of (30), which means that the lefthand side of (30) is nonnegative, a contradiction. The only remaining option is $d = 1$. In this case, d is larger than 2 of the zeros of the lefthand side of (30) (counted with multiplicity), but d is less than or equal to the other two zeros. So, the lefthand side of (30) is again nonnegative, a contradiction.

Next, suppose the sign of the square roots in (30) is negative. Then, both factors of the lefthand side of (30) go to $-\infty$ as $d \rightarrow \infty$, and none of the possible zeros of the lefthand side of (30) is an actual zero. This implies that the lefthand side of (30) is always positive, a contradiction. \square

3.3 Proof of Theorem 1.2

Throughout this proof, the symbol ' \leq ' between two convex generators means ' $\leq_{P(a,1),B(c)}$ '.

Proof of Theorem 1.2. Suppose by way of contradiction that $c < 2 + a/2$ and that $P(a, 1)$ symplectically embeds into $B(c)$. By Proposition 3.2, there exists some d_a such that for any $d > d_a$, there is no convex generator Λ satisfying $\Lambda \leq e_{1,1}^d$. For any $d \in \mathbb{Z}_{>0}$, define,

$$N_d = \{\Lambda \mid \Lambda \leq e_{1,1}^d\},$$

and let,

$$N = \sum_{d=1}^{d_a} dN_d.$$

Note that for any d , there are a finite number of convex generators with index equal to $I(e_{1,1}^d)$, which implies that the N_d and N are finite.

Now, fix any integer $D > N$. The generator $\Lambda' = e_{1,1}^D$ is minimal for $B(c)$ by [8, Lemma 2.1]. So, we can apply Theorem 1.14 to obtain a convex generator Λ , an integer n , and factorizations $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$ satisfying the three numbered conditions of Theorem 1.14.

Suppose there exists some $i \neq j$ such that $\Lambda'_i = \Lambda'_j$ and $\Lambda_i = \Lambda_j$. Then, let $\Gamma = \Lambda_i = \Lambda_j$, and write $\Lambda'_i = \Lambda'_j = e_{1,1}^d$ for some d . Condition (i) of Theorem 1.14 implies that $\Gamma \leq e_{1,1}^d$, and condition (iii) of Theorem 1.14 implies,

$$I(\Gamma \cdot \Gamma) = I(\Lambda'_i \cdot \Lambda'_j) = I(e_{1,1}^{2d}).$$

However, Γ and d then contradict the statement of Proposition 3.3. So, for all $i \neq j$, we must have either $\Lambda'_i \neq \Lambda'_j$ or $\Lambda_i \neq \Lambda_j$.

We claim that with this constraint, it is impossible to have $I(\Lambda') = I(\prod_{i=1}^n \Lambda'_i)$. Since $N_d = 0$ for all $d > d_a$, the maximum possible value of $I(\prod_{i=1}^n \Lambda'_i)$ occurs when, for any $d \leq d_a$ and any $\eta \leq e_{1,1}^d$, there is precisely one choice of i such that $\Lambda'_i = e_{1,1}^d$ and $\Lambda_i = \eta$. In this case, we get,

$$I\left(\prod_{i=1}^n \Lambda'_i\right) = I\left(\prod_{d=1}^{d_a} \prod_{i=1}^{N_d} e_{1,1}^d\right) = \left(e_{1,1}^{\sum_{d=1}^{d_a} dN_d}\right) = I(e_{1,1}^N) = N(N+3).$$

However,

$$I(\Lambda') = I(e_{1,1}^D) = D(D+3) > N(N+3),$$

since $D > N$. Any other choice of the Λ_i must be a subset of this choice of Λ_i , in which case $I(\prod_{i=1}^n \Lambda'_i)$ will be even smaller. Thus, there are no possible choices for the Λ_i such that $I(\prod_{i=1}^n \Lambda'_i) = I(\Lambda')$, contradicting the fact that $\prod_{i=1}^n \Lambda' = \Lambda'$. \square

3.4 Difficulties extending Theorem 1.2 via the Hutchings criterion

Theorem 1.2 implies that symplectic folding yields optimal embeddings of $P(a, 1)$ into $B(c)$ whenever,

$$2 \leq a < \frac{\sqrt{7}-1}{\sqrt{7}-2} = 2.54858\dots$$

For $a \geq \frac{\sqrt{7}-1}{\sqrt{7}-2}$, our method of proving Theorem 1.2 breaks down. More specifically, the proof of Proposition 3.2 relies on the fact that $a < \frac{\sqrt{7}-1}{\sqrt{7}-2}$ in order to conclude that the coefficient of d^2 in (27) is positive, yielding a contradiction for sufficiently large d . When a is larger than this value, the conclusions of the proposition will no longer hold, so we will no longer be able to consider convex generators $e_{1,1}^d$ for arbitrarily large d in the proof of Theorem 1.2.

It is natural to ask whether this upper bound on a can be extended by applying the Hutchings criterion and using different methods of proof than those used in Theorem 1.2. Since $e_{1,1}^d$ is a minimal generator for $B(c)$ for all $d \geq 1$, we might try applying the Hutchings criterion to $e_{1,1}^d$ for some specific, not necessarily large choice of d , allowing us to avoid the use of Proposition 3.2. We would then argue as follows. For some fixed $a \geq \frac{\sqrt{7}-1}{\sqrt{7}-2}$, suppose we have some $c < 2 + a/2$ such that $P(a, 1)$ symplectically embeds into $B(c)$. We can apply the modified Hutchings criterion, Theorem 1.14, to $\Lambda' = e_{1,1}^d$ to obtain an integer n , a convex generator Λ , and factorizations $\Lambda' = \Lambda'_1 \cdots \Lambda'_n$ and $\Lambda = \Lambda_1 \cdots \Lambda_n$.

To obstruct the symplectic embedding we assumed to exist, we must show that no possible choice of the Λ_i and Λ'_i exists. In particular, we must show that there exists no convex generator Γ such that $\Gamma \leq_{P(a,1), B(c)} e_{1,1}^d$: otherwise, we will not be able to obstruct the possibility that $n = 1$, $\Lambda'_1 = \Lambda' = e_{1,1}^d$, and $\Lambda_1 = \Lambda = \Gamma$.

However, we can actually prove that for any $a \geq \frac{\sqrt{7}-1}{\sqrt{7}-2}$ and any $d \geq 1$, there is some $c < 2 + a/2$ and some convex generator Γ such that $\Gamma \leq_{P(a,1), B(c)} e_{1,1}^d$ for every $d \geq 1$. This implies that it is impossible to improve on the results of Theorem 1.2 by applying the Hutchings criterion to convex generators of the form $e_{1,1}^d$. In order to prove this fact, we must first construct a convex generator satisfying certain constraints.

Lemma 3.4. *Let $d \geq 9$. Then, there exists some convex generator $\Lambda = Fe_{1,0}e_{m,1}Ve_{0,1}$ such that,*

$$0 \leq F \leq \frac{1}{2} \left(3d - 1 + \sqrt{7d^2 - 3} \right), \quad (31)$$

$$V = \frac{1}{2} \left(3d - 2 - \sqrt{7d^2 - 6d + 4F} \right), \quad (32)$$

and,

$$m = \frac{1}{2} \left(3d - 2 + \sqrt{7d^2 - 6d + 4F} \right) - F. \quad (33)$$

Proof. Suppose we have some choice of F , V , and m that satisfies (31), (32), and (33). Notice that $V \geq 0$ whenever,

$$(3d - 2)^2 \geq 7d^2 - 6d + 4F,$$

i.e. whenever,

$$\frac{1}{2}(d^2 - 3d + 2) \geq F. \quad (34)$$

On the other hand, using (31) and the fact that $d \geq 9$, we have,

$$F \leq \frac{1}{2}(3d - 1 + \sqrt{7d^2 - 3}) \leq \frac{1}{2}(d^2 - 3d + 2),$$

so that (34) is true and $V \geq 0$. Similarly, $m \geq 0$ whenever,

$$\sqrt{7d^2 - 6d + 4F} \geq 2F + 2 - 3d,$$

or equivalently whenever,

$$F \leq \frac{1}{2} \left(3d - 1 + \sqrt{7d^2 - 3} \right).$$

This inequality is true by (31), so we must have $m \geq 0$.

Since V and m are necessarily nonnegative, it remains to find some F satisfying (31) such that the definitions of V and m in (32) and (33) are integers. Assuming that $\sqrt{7d^2 - 6d + 4F}$ is an integer, this square root will be even if and only if d is even, which implies that V and m will both be integers no matter what the parity of d is. So, it suffices to show that we can pick $\sqrt{7d^2 - 6d + 4F}$ to be an integer.

Let k^2 be the largest perfect square less than $7d^2$. Then we have,

$$7d^2 - C = k^2 \quad (35)$$

for some $C > 0$. Because $7d^2 < (k + 1)^2$, the above equation yields,

$$C = 7d^2 - k^2 < (k + 1)^2 - k^2 = 2k + 1 = 2\sqrt{7d^2 - C} + 1. \quad (36)$$

The righthand side of this inequality is less than $6d$ whenever,

$$(6d - 1)^2 > 28d^2 - 4C,$$

or equivalently whenever,

$$8d^2 - 12d + 4C + 1 > 0. \quad (37)$$

The discriminant of the lefthand side of this inequality is,

$$144 - 128C - 32 = 112 - 128C,$$

which is negative because $C \geq 1$. So, (37) is true, which means that the righthand side of (36) is less than $6d$. We then obtain,

$$C < 6d. \quad (38)$$

There are now 3 cases to consider, depending on the residue class of C modulo 4.

1. Suppose that $C \equiv 1 \pmod{4}$. Taking (35) mod 4 gives us,

$$-d^2 - 1 \equiv k^2 \pmod{4},$$

or equivalently,

$$3 \equiv k^2 + d^2 \pmod{4}.$$

However, the only squares mod 4 are 0 and 1, so this is impossible. Thus, we cannot have $C \equiv 1 \pmod{4}$.

2. Suppose that $C \equiv 3 \pmod{4}$. Then, we define,

$$F = \frac{1}{2} \left(3d - \frac{C-1}{2} + \sqrt{7d^2 - C} \right).$$

Because $C \geq 3$, this choice of F satisfies the upper bound on F given by (31), and (38) gives us,

$$F > \frac{1}{2} \left(3d - \frac{C}{2} \right) = \frac{6d - C}{4} > 0,$$

so that the lower bound of (31) is also satisfied. Moreover, (35) and the fact that C is odd imply that d and $\sqrt{7d^2 - C} = k$ have opposite parity. So, no matter what the parity of d is, F is an integer. Finally, we have,

$$\sqrt{7d^2 - 6d + 4F} = \sqrt{7d^2 - C + 2\sqrt{7d^2 - C} + 1} = \sqrt{\left(\sqrt{7d^2 - C} + 1\right)^2} = k + 1,$$

which is an integer, as desired.

3. Suppose that $C \equiv 0, 2 \pmod{4}$. Then we pick,

$$F = \frac{6d - C}{4}.$$

Notice that $F > 0$ by (38) while,

$$F \leq \frac{1}{2}(3d - 1 + \sqrt{7d^2 - 3})$$

is equivalent to,

$$1 - C \leq \sqrt{7d^2 - 3} = k,$$

which is true by definition of C . Thus, (31) is satisfied. Moreover, taking (35) mod 4 gives,

$$-d^2 - C \equiv k^2 \pmod{4},$$

or equivalently,

$$-C \equiv k^2 + d^2 \pmod{4}. \tag{39}$$

If $C \equiv 2 \pmod{4}$, then (39) implies that $k^2 \equiv d^2 \equiv 1 \pmod{4}$, so d must be odd. In this case, $6d - C \equiv 0 \pmod{4}$, whence our choice of F is an integer. Likewise, if $C \equiv 0 \pmod{4}$, then (39) implies that $k^2 \equiv d^2 \equiv 0 \pmod{4}$, so d must be even. In this case, both $6d$ and C are divisible by 4, so F is again an integer. Finally, we have,

$$\sqrt{7d^2 - 6d + 4F} = \sqrt{7d^2 - C} = k,$$

which is an integer, as desired. □

We now use the above lemma to prove that applying the Hutchings criterion to $e_{1,1}^d$ for any $d \geq 1$ cannot improve upon Theorem 1.2.

Proposition 3.5. *Let*

$$a \geq \frac{\sqrt{7} - 1}{\sqrt{7} - 2} = 2.54858\dots$$

For any $d \geq 1$, there exists some $\epsilon > 0$ and some convex generator Λ such that $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$, where $c = 2 + a/2 - \epsilon$.

Proof. First, note that when $d = 1$, we have $e_{1,0}^2 \leq_{P(a,1),B(c)} e_{1,1}$ for any $c \geq 2$, and when $d = 2$, we have $e_{1,0}^5 \leq_{P(a,1),B(c)} e_{1,1}^2$ for any $c \geq 2.5$. Since 2 and 2.5 are less than $2 + a/2$ for any possible value of a , the desired statement follows for $d = 1, 2$. Moreover, if $3 \leq d \leq 8$, we can define $\Lambda = Fe_{1,0}e_{m,1}$ where,

$$F = \frac{1}{2}(d^2 - 3d + 2),$$

and,

$$m = \frac{1}{2}(-d^2 + 9d - 6).$$

F and m are positive integers for all $3 \leq d \leq 8$. In addition we have,

$$\begin{aligned} x(\Lambda) + y(\Lambda) &= F + m + 1 \\ &= \frac{1}{2}(6d - 4) + 1 \\ &= 3d - 1 \\ &= x(e_{1,1}^d) + y(e_{1,1})^d + m(e_{1,1}^d) - 1, \end{aligned} \tag{40}$$

and (3) yields,

$$\begin{aligned} I(\Lambda) &= 2F + m + 2F + m + 2 \\ &= 2d^2 - 6d + 4 - d^2 + 9d - 6 + 2 \\ &= d^2 + 3d \\ &= I(e_{1,1}^d). \end{aligned} \tag{41}$$

Finally,

$$A_{P(a,1)}(\Lambda) = x(\Lambda) + ay(\Lambda) = 3d - 2 + a,$$

so that $A_{P(a,1)}(\Lambda) < (2 + a/2)d$ whenever,

$$a > \frac{2(d-2)}{d-2} = 2.$$

Because $a > 2$ by assumption, we must have $A_{P(a,1)}(\Lambda) < (2 + a/2)d$. Then, for any $0 < \epsilon \leq (2 + a/2) - A_{P(a,1)}(\Lambda)/d$, we obtain

$$A_{P(a,1)}(\Lambda) \leq (2 + a/2 - \epsilon)d = A_{B(2+a/2-\epsilon)}(e_{1,1}^d).$$

This equation along with (40) and (41) implies that $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$ for $c = 2 + a/2 - \epsilon$, as desired.

We are left with the case where $d \geq 9$. Here, we can apply Lemma 3.4 to construct some convex generator $\Lambda = Fe_{1,0}e_{m,1}Ve_{0,1}$ satisfying (31), (32), and (33). We will prove that $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$ for some c of the desired form. First, notice that,

$$\begin{aligned} x(\Lambda) + y(\Lambda) &= (F + m) + (V + 1) \\ &= \frac{1}{2} \left(3d - 2 - \sqrt{7d^2 - 6d + 4F} \right) + \frac{1}{2} \left(3d - 2 + \sqrt{7d^2 - 6d + 4F} \right) + 1 \\ &= 3d - 1 \\ &= x(e_{1,1}^d) + y(e_{1,1}^d) + m(e_{1,1}^d) - 1. \end{aligned} \tag{42}$$

Moreover, using (3) and substituting in (42) gives,

$$\begin{aligned} I(\Lambda) &= 2\mathbb{A}(\Lambda) + x(\Lambda) + y(\Lambda) + m(\Lambda) \\ &= 2F(V + 1) + m(2V + 1) + 3d - 1 + F + V + 1 \\ &= 2F(V + 1) + 2Vm + 3d - 1 + F + V + m + 1 \\ &= 2V(F + m) + 2F + 3d - 1 + x(\Lambda) + y(\Lambda) \end{aligned}$$

Substituting in (42) again and using the definitions of m and V produces,

$$\begin{aligned} I(\Lambda) &= \frac{1}{2} \left(3d - 2 - \sqrt{7d^2 - 6d + 4F} \right) \left(3d - 2 + \sqrt{7d^2 - 6d + 4F} \right) + 2F + 6d - 2 \\ &= 2 - 3d + d^2 - 2F + 2F + 6d - 2 \\ &= d^2 + 3d = I(e_{1,1}^d) \end{aligned} \tag{43}$$

In light of (43) and (42), we see that $\Lambda \leq_{P(a,1),B(c)} e_{1,1}^d$ if and only if $A_{P(a,1)}(\Lambda) \leq A_{B(c)}(e_{1,1}^d)$. We will show,

$$A_{P(a,1)}(\Lambda) < \left(2 + \frac{a}{2} \right) d. \tag{44}$$

Then, for any $0 < \epsilon \leq (2 + a/2) - A_{P(a,1)}(\Lambda)/d$, we have,

$$A_{P(a,1)}(\Lambda) \leq \left(2 + \frac{a}{2} - \epsilon\right) d = A_{B(2+a/2-\epsilon)}(e_{1,1}^d),$$

so that $\Lambda \leq_{P(a,1), B(c)} e_{1,1}^d$, where $c = 2 + a/2 - \epsilon$.

To prove (44), we first use (42) and (32) to compute $A_{P(a,1)}(\Lambda)$:

$$\begin{aligned} A_{P(a,1)}(\Lambda) &= x(\Lambda) + ay(\Lambda) = 3d - 1 + (a - 1)y(\Lambda) \\ &= 3d - 1 + (a - 1) \left(\frac{1}{2} \left(3d - 2 - \sqrt{7d^2 - 6d + 4F} \right) + 1 \right) \end{aligned}$$

Using this calculation, (44) is equivalent to,

$$(a - 1) \left(3d - \sqrt{7d^2 - 6d + 4F} \right) < (a - 2)d + 2.$$

Rearranging produces,

$$\sqrt{7d^2 - 6d + 4F} - d - 2 < a \left(\sqrt{7d^2 - 6d + 4F} - 2d \right).$$

Since $\sqrt{7d^2 - 6d + 4F} - 2d \geq \sqrt{7d^2 - 6d} - 2d > 0$ for all $d > 2$, the above inequality becomes,

$$\frac{\sqrt{7d^2 - 6d + 4F} - d - 2}{\sqrt{7d^2 - 6d + 4F} - 2d} < a. \quad (45)$$

The lefthand side of (45) is increasing for all F and all $d > 2$, and its limit as $d \rightarrow \infty$ is

$$\frac{\sqrt{7} - 1}{\sqrt{7} - 2} = 2.54858 \dots$$

Since a is at least this limit value by assumption and $d \geq 9$, we conclude that (45) is true, hence so is (44). \square

3.5 Trying other convex generators

Now that we know we cannot use any generator of the form $e_{1,1}^d$ to improve upon the results of Theorem 1.2, we might ask if we can apply the Hutchings criterion to any other minimal generator for the ball. However, minimal generators must *uniquely* minimize the symplectic action among all convex generators of equal index. The following lemma shows that in every index grading other than those of the $e_{1,1}^d$, the action with respect to any ball is non-uniquely minimized, so that the $e_{1,1}^d$ are the only minimal generators for $B(c)$.

Lemma 3.6. *Let $c > 0$, and let k be a positive integer such that $2k \neq I(e_{1,1}^d)$ for all $d \geq 1$. Then, there exist two distinct convex generators which minimize the symplectic action with respect to $B(c)$ among convex generators with index $2k$.*

Proof. The proof is by construction. Let d be the smallest positive integer such that $I(e_{1,1}^d) > 2k$, and let $\delta = I(e_{1,1}^d)/2 - k$. We construct a finite sequence of convex generators $Y_1, Y_2, \dots, Y_\delta$ by induction. In the base case, set $Y_1 = e_{1,0}e_{1,1}^{d-1}$. For all $i \geq 2$, define Y_i from Y_{i-1} according to the following rules.

1. If $Y_{i-1} = e_{1,0}^a e_{1,1}^m$ for some a and m , then $Y_i = e_{1,0}^{a-1} e_{2,1} e_{1,1}^{m-1}$ if $a > 1$, and $Y_i = e_{2,1} e_{1,1}^{m-1}$ if $a = 1$.
2. If $Y_{i-1} = e_{1,0}^a e_{b,1} e_{1,1}^m$ for some a, b , and m , then $Y_i = e_{1,0}^{a-1} e_{b+1,1} e_{1,1}^m$ if $a > 1$, and $Y_i = e_{b+1,1} e_{1,1}^m$ if $a = 1$.
3. If $Y_{i-1} = e_{a,1} e_{1,1}^m$ for some a and m , then $Y_i = e_{1,0}^{d-m} e_{1,1}^m$.

Conceptually, Y_1 is equal to $e_{1,1}^d$ but with the uppermost lattice point removed, and in general, Y_i is equal to Y_{i-1} with one lattice point removed. As an example, the first three Y_i when $d = 3$ are shown in Figure 2.

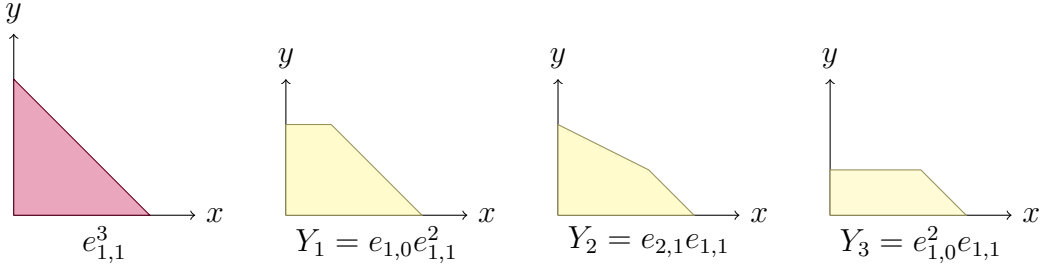


Figure 2: The first few Y_i when $d = 3$. The generator $e_{1,1}^d$ is shown on the left for comparison. Note that every generator is the same as the one to the left but with one lattice point removed.

By construction, we have $I(Y_1) = I(e_{1,1}^d) - 2$ and $I(Y_i) = I(Y_{i-1}) - 2$ for all $i \geq 2$. This implies that $I(Y_i) = I(e_{1,1}^d) - 2i$ for all i and in particular that $I(Y_\delta) = 2k$. We claim that Y_δ minimizes the symplectic action with respect to $B(c)$ among all convex generators with index $2k$.

To this end, note that for any convex generator Λ with $I(\Lambda) = 2k$, we have $A_{B(c)}(\Lambda) = c(m+n)$, where (m,n) is the vertex of Λ at which a line of slope -1 is tangent. Now, $m+n$ is the y -intercept of the line of slope -1 through (m,n) . For any other vertex (a,b) of Λ , the line of slope -1 through (a,b) is not tangent to Λ and so has strictly smaller y -intercept than the tangent line of slope -1 . This implies that $m+n \geq a+b$ for any vertex (a,b) of Λ , with equality if and only if $(m,n) = (a,b)$.

Now, $I(\Lambda) = 2k > I(e_{1,1}^{d-1})$ by the definition of d , so we know that Λ contains some lattice point (a,b) not contained in $e_{1,1}^{d-1}$. Using our above arguments, we then have,

$$A_{B(c)}(\Lambda) = c(m+n) \geq c(a+b) > c(d-1),$$

so that in fact, $A_{B(c)}(\Lambda) \geq cd$. On the other hand, the line $x + y = d$ is tangent to Y_i for all i by construction, which implies that $A_{B(c)}(Y_i) = cd$. In particular, we obtain,

$$A_{B(c)}(Y_\delta) = cd \leq A_{B(c)}(\Lambda),$$

as desired.

Next, define X_δ to be the reflection of Y_δ about the line $y = x$. The line $x + y = d$ is tangent to X_δ , so we have $A_{B(c)}(X_\delta) = A_{B(c)}(Y_\delta) = cd$. This implies that X_δ also minimizes the symplectic action of $B(c)$ among convex generators with index $2k$. Finally, we note that $X_\delta \neq Y_\delta$ because Y_δ is not symmetric about the line $y = x$. \square

The above lemma tells us that we cannot apply Theorem 1.14 to any convex generators other than the $e_{1,1}^d$ in order to understand symplectic embeddings into the ball. Combined with Theorem 3.5, this implies that in fact, Theorem 1.14 cannot be used to extend the upper bound on a in the statement of Theorem 1.2.

The conjectural improvement of the Hutchings criterion [8, Conj. A.3], proven in [1], allows the statement of Theorem 1.14 to be weakened so that one need only assume that all edges of Λ' are labelled ‘ e ’ (as opposed to the requirement that Λ' be minimal). instead of not that the generator is minimal. As a result it still might be possible to improve upon Theorem 1.2 using a non-minimal generator.

For instance, we could try to apply the Hutchings criterion to the convex generators constructed in Lemma 3.6, which non-uniquely minimize the symplectic action in their index grading. However, preliminary evidence suggests that these generators (as well as all others of equal index and symplectic action) will do no better than the $e_{1,1}^d$.

Moreover, [8, Conj. A.3] would also allow one to use a generator that does not minimize the symplectic action at all. This choice would likely weaken the action inequality in the definition of ‘ \leq ’ between convex generators for most relevant cases. Thus the Hutchings criterion should on the whole yield weaker combinatorial conditions for non-minimal generators than it does for minimal ones. In short, some possibility remains to extend the statement of Theorem 1.2 to larger values of a using the Hutchings criterion, but it will require methods beyond the scope of this paper.

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